RAY STATISTICS IN A RANDOM MEDIUM WITH STRATIFIED BACKGROUND

by

C. H. LIU

DIFFUSION OF RAYS IN RANDOM MEDIA

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C. H. LIU and K. C. YEH

February, 1968

Sponsored by
National Aeronautics and Space Administration
Washington, D. C. 20546
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Technical Report No. 33

Ionosphere Radio Laboratory

ECTRICAL ENGINEERING RESEARCH LABORATORY
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RAY STATISTICS IN A RANDOM MEDIUM WITH STRATIFIED BACKGROUND

C. H. Liu

Abstract

The problem of propagation of light rays in a stratified medium in the presence of random irregularities is investigated. Starting from the ray equations, a systematic asymptotic expansion is used to find the ray trajectory. Statistical quantities of the ray such as the mean square fluctuations of the transverse displacement and the ray direction are then computed. The results are compared with those for a homogeneous background.

1. Introduction

When a light beam propagates in a turbulent medium, the random variation of the refractive index of the medium will cause random fluctuations in the propagation parameters of the beam in many ways. The most important ones are the fluctuations in the angle of arrival, in the position of the beam, in the cross section of the beam and in the intensity of the beam. These quantities are useful in designing an optical system involving propagation through turbulent medium. Many authors have studied this problem before with the assumption that the medium is weakly random and the background is homogeneous (Chernov, 1960; Keller, 1962; Bremmer, 1964). In many practical cases, however, situations are quite different. For example, when one uses laser beams to diagnose a turbulent plasma, the background plasma density may well be inhomogeneous in addition to the random variation of the turbulent region. Therefore, it is of both theoretical and experimental interests to study the statistics of light rays in a random medium with inhomogeneous background. In this paper, we shall try to take the first step in this direction.

The background medium is taken as a stratified one in which the refractive index varies in only one direction, say, z. A light ray is assumed to start at the origin in the z-direction. The more general case of a ray with an arbitrary incident angle will be treated separately. The random variation of the refractive index will cause the ray to fluctuate from its original direction of propagation. This causes the ray to displace transversely. The statistics of the transverse displacement shall be shown to be related to the properties of the medium.

In Section 2, a systematic asymptotic expansion is used to solve the ray equation for an almost-stratified medium. Results are then applied to the random medium case in Sec. 3 to compute the mean position of the ray, the mean square fluctuations of the total displacement, the transverse displacement and the ray direction. In Sec. 4, explicit computations are made for a special example and results are compared with those derived by Keller (1962) for the case of homogeneous background. Some concluding remarks are given in Sec. 5.

2. Light rays in an almost-stratified medium

Let $n(\vec{x},\epsilon)$ denote the index of refraction of a medium, which is written as

$$n(\vec{x},\epsilon) = n_o(z) + \epsilon\mu(\vec{x}) \tag{1}$$

This medium is said to be almost stratified in the sense that $n(\vec{x},\epsilon)$ varies mainly in one direction, namely the z direction. The small parameter ϵ measures the random deviation of the index from the case of exact stratification. Both $n_o(z)$ and the statistical properties of μ are assumed given. The problem is to study the ray $\vec{x}(s,\epsilon)$ which starts at the origin in the direction of z-axis. Here s denotes the arc length along the ray path. The equation that determines the ray can be written

$$\frac{d}{ds}\left(n\frac{d\vec{x}}{ds}\right) = \nabla n(\vec{x},\epsilon) \tag{2}$$

which together with initial conditions

$$\vec{x}(0) = 0 \tag{3}$$

$$\frac{d\vec{x}}{ds}(0) = \hat{k} \tag{4}$$

completely specify the problem at hand.

To solve the equations, the following expansion is used (Keller, 1962):

$$\vec{x}(s,\epsilon) = \vec{x}_o(s) + \epsilon\vec{x}_\epsilon(s) + \epsilon^2\,\vec{x}_{\epsilon\epsilon}(s) + 0(\epsilon^3) \tag{5}$$

The quantities $n(\vec{x},\epsilon)$ and $\nabla n(\vec{x},\epsilon)$ are then Taylor expanded about $\vec{x} = \vec{x}_o$. Substituting the results into (2) to (4) and equate equal power terms in ϵ, equations for $\vec{x}_o$, $\vec{x}_\epsilon$, $\vec{x}_{\epsilon\epsilon}$ can be obtained. In the following, terms of the order higher than ϵ^2 are neglected.

The zeroth order equations are given by

$$\frac{d}{ds}\left[n_o(z_o)\,\frac{d\vec{x}_o(s)}{ds}\right] = \frac{dn_o}{dz}(z_o)\;\hat{k} \tag{6}$$

$$\vec{x}_o(0) = 0 \tag{7}$$

$$\frac{d\vec{x}_o}{ds}(0) = \hat{k} \tag{8}$$

The solution of (6)-(8) is

$$\vec{x}_o(s) = s\,\hat{k} \tag{9}$$

The first order equations can be written in component forms

$$\frac{d}{ds}\left[n_o(z_o)\,\frac{dx_\epsilon(s)}{ds}\right] = \frac{\partial\mu}{\partial x}(\vec{x}_o) \tag{10}$$

$$\frac{d}{ds}\left[n_o(z_o)\,\frac{dy_\epsilon(s)}{ds}\right] = \frac{\partial\mu}{\partial y}\,(\vec{x}_o) \tag{11}$$

$$\frac{d}{ds}\left[n_o(z_o)\,\frac{dz_\epsilon(s)}{ds} + z_\epsilon(s)\,\frac{dn_o}{dz}\,(z_o)\right] - \frac{d^2 n_o}{dz^2}\,(z_o) z_\epsilon(s) = 0 \tag{12}$$

$$\vec{x}_\epsilon(0) = \big(x_\epsilon(0),\ y_\epsilon(0),\ z_\epsilon(0)\big) = 0 \tag{13}$$

$$\frac{d\vec{x}_\epsilon}{ds}\,(0) = 0 \tag{14}$$

From (12), (13) and (14), we obtain

$$z_\epsilon(s) = 0 \tag{15}$$

Equations (10), (11), (13), and (14) yield

$$\vec{x}_\epsilon(s) = \int_0^s \left[\int_t^s \frac{1}{n_o(\tau)}\,d\tau\right] \nabla_T\mu(t\hat{k})\,dt \tag{16}$$

where ∇_T is the gradient transverse to the $\hat{z}$-axis, the original direction of the ray.

The second order equations, expressed in component forms, are

$$\frac{d}{ds}\left[n_o(z_0)\,\frac{dx_{\epsilon\epsilon}(s)}{ds}\right] = [\vec{x}_\epsilon(s)\cdot\nabla_T]\,\frac{\partial\mu}{\partial x}\,(\vec{x}_o) - \frac{d}{ds}\left[\mu(\vec{x}_o)\,\frac{dx_\epsilon(s)}{ds}\right] \tag{17}$$

$$\frac{d}{ds}\left[n_o(z_o)\,\frac{dy_{\epsilon\epsilon}(s)}{ds}\right] = [\vec{x}_\epsilon(s)\cdot\nabla_T]\,\frac{\partial\mu}{\partial y}\,(\vec{x}_o) - \frac{d}{ds}\left[\mu(\vec{x}_o)\,\frac{dy_\epsilon(s)}{ds}\right] \tag{18}$$

$$\frac{d}{ds}\left[n_o(z_o)\,\frac{dz_{\epsilon\epsilon}(s)}{ds} + \frac{dn_o}{dz}\,(z_o)\,z_{\epsilon\epsilon}(s)\right] - \frac{d^2 n_o}{dz^2}\,(z_o)\,z_{\epsilon\epsilon}(s)$$

$$= -\frac{d\vec{x}_\epsilon(s)}{ds}\cdot\nabla_T\mu(\vec{x}_o) \tag{19}$$

$$\vec{x}_{\epsilon\epsilon}(0) = (x_{\epsilon\epsilon}(0),\ y_{\epsilon\epsilon}(0),\ z_{\epsilon\epsilon}(0)) = 0 \tag{20}$$

$$\frac{d\vec{x}_{\epsilon\epsilon}}{ds}(0) = 0 \tag{21}$$

Solution of (17)-(21) is

$$\vec{x}_{\epsilon\epsilon}(s) = \int_0^s dt \left[\int_t^s \frac{1}{n_o(\tau)} d\tau\right] \left[(\vec{x}_\epsilon(t)\cdot\nabla_T)\nabla_T\mu(t\hat{k}) - \frac{1}{2n_o} \nabla_T\mu^2(t\hat{k})\right.$$

$$\left. + \mu(t\hat{k}) \frac{1}{n_o^2(t)} \frac{dn_o}{dt} \int_0^t \nabla_T\mu(\tau\hat{k})d\tau - \hat{k}\cdot\nabla\mu(t\hat{k}) \int_0^t \frac{1}{n_o(t)}\nabla_T\mu(\hat{k}\tau)d\tau\right]$$

$$- \hat{k} \int_0^s dt \left[\int_t^s \frac{n_o(t)}{n_o^2(\tau)} d\tau\right] \nabla_T\mu(\hat{k}t)\cdot\int_0^t \frac{1}{n_o(t)} \nabla_T\mu(\hat{k}\tau)d\tau \tag{22}$$

Equations (5), (9), (16) and (22) determine the ray as a function of arc length along the ray path up to ϵ^2 terms.

3. Light rays in a random medium with stratified background

With the derivation in the last section, we are ready to discuss the problem of light rays in a random medium with stratified background. If $\mu(\vec{x})$ in Eq. (1) in the expression for refractive index is assumed to be a random function of position with the following statistical properties

$$< \mu(\vec{x}) > = 0 \tag{23}$$

$$< \mu(\vec{x}_1)\mu(\vec{x}_2) > = < \mu^2 > N\left(\left|\vec{x}_1 - \vec{x}_2\right|\right) \tag{24}$$

where $< >$ denotes average, $< \mu^2 >$ is a constant, then it is possible to compute some statistical properties of the ray $\vec{x}(s,\epsilon)$.

From eq. (5), we have

$$< \vec{x}(s,\epsilon) > \; = \; < \vec{x}_o(s) > + \epsilon < \vec{x}_\epsilon(s) > + \epsilon^2 < \vec{x}_{\epsilon\epsilon}(s) > + 0(\epsilon^3)$$

$$= s\,\hat{k} + \epsilon^2 < \vec{x}_{\epsilon\epsilon}(s) > + 0(\epsilon^3) \tag{25}$$

where eq. (23) has been used.

From eq. (22), we have:

$$\begin{aligned}
<\vec{x}_{\epsilon\epsilon}> = \int_0^s dt_2 \left[\int_{t_2}^{s} \frac{d\tau}{n_o(\tau)}\right] \Bigg\{ \int_0^{t_2} dt_1 \left[\int_{t_1}^{t_2} \frac{d\tau}{n_o(\tau)}\right] < \nabla_{T_1}\mu(t_1\hat{k}) \cdot \nabla_{T_2}\nabla_{T_2}\mu(t_2\hat{k}) > \\
- \frac{1}{2n_o(t_2)} < \nabla_{T_2}\mu^2(t_2\hat{k}) > + \frac{1}{n_o^2(t_2)} \frac{dn_o}{dt_2} < \mu(t_2\hat{k}) \int_0^{t_2} \nabla_{T_1}\mu(t_1\hat{k})dt_1 > \\
- \frac{1}{n_o(t_2)}\, \hat{k} \cdot <\nabla\mu(t_2\hat{k}) \int_0^{t_2} \nabla_{T_1}\mu(t_1\hat{k})dt_1 > \Bigg\} \\
- \hat{k} \int_0^s dt_2 \left[\int_{t_2}^{s} \frac{n_o(t_2)}{n^2(\tau)}\, d\tau\right] <\nabla_{T_2}\mu(t_2\hat{k}) \cdot \int_0^{t_2} \frac{1}{n_o(t_2)} \nabla_{T_1}\mu(\hat{k}t_1)dt_1 >
\end{aligned} \tag{26}$$

We have assumed that it is permissible to interchange the order of taking the mean value with integration and differentiation. We first consider the averaged quantity in the last term

$$< \nabla_{T_2}\mu(t_2\hat{k})\nabla_{T_1}\mu(t_1\hat{k}) > = < \nabla_{T_1}\mu(\vec{x}_1)\nabla_{T_2}\mu(\vec{x}_2) > \Big|_{\vec{x}_1 = t_1\hat{k},\ \vec{x}_2 = t_2\hat{k}}$$

$$= \nabla_{T_1}\cdot\nabla_{T_2} < \mu(\vec{x}_1)\mu(\vec{x}_2) > \Big|_{\vec{x}_1 = t_1\hat{k},\ \vec{x}_2 = t_2\hat{k}}$$

$$= < \mu^2 > \nabla_{T_1}\cdot\nabla_{T_2} N(|\vec{x}_1-\vec{x}_2|)\Big|_{\vec{x}_1=t_1\hat{k},\ \vec{x}_2=t_2\hat{k}} \quad (27)$$

Since N is a function of $|\vec{x}_1 - \vec{x}_2|$ only, therefore $\nabla_{T_1} = -\nabla_{T_2}$ and (27) becomes

$$- < \mu^2 > \nabla^2_{T_2} N(|\vec{x}_1 - x_2|)\Big|_{\vec{x}_1 = t_1\hat{k},\ \vec{x}_2 = t_2\hat{k}} = - \frac{2}{r}\frac{\partial N(r)}{\partial r} \quad (28)$$

where

$$r = \left|t_1 - t_2\right| \quad (29)$$

By the same procedure, it is easy to show that all the other terms in (26) are zero, therefore

$$< \vec{x}_{\epsilon\epsilon} > = 2\hat{k} < \mu^2 > \int_0^s dt_2\left[\int_{t_2}^s \frac{n_o(t_2)}{n_o^2(\tau)}\,d\tau\right] \int_0^{t_2}\frac{1}{n_o(t_2)}\,\frac{1}{r}\,N_r(r)dt_1$$

$$= 2\,\hat{k} < \mu^2 > \int_0^s \frac{N_r(r)}{r}\,dr \int_r^s\left[\int_{t_2}^s \frac{d\tau}{n_o^2(\tau)}\right] dt_2 \quad (30)$$

$$<\vec{x}(s,\epsilon)> = \hat{k}[s + 2\epsilon^2 < \mu^2 > \int_0^s \frac{N_r(r)}{r}dr \int_r^s\left[\int_{t_2}^s \frac{d\tau}{n_o^2(\tau)}\right] dt_2] + 0(\epsilon^3) \quad (31)$$

Eq. (31) gives the mean position of the end point of a ray of length s. The average ray to the order of ϵ^2 is still in the original direction.

Next, let us compute some important mean square values. From (5)

$$\vec{x}(s,\epsilon)\cdot\vec{x}(s,\epsilon) = s^2 + 2\epsilon s\hat{k}\cdot\vec{x}_\epsilon + \epsilon^2[\vec{x}_\epsilon\cdot\vec{x}_\epsilon + 2s\,\vec{x}_{\epsilon\epsilon}\cdot\hat{k}] + 0(\epsilon^3)$$

$$= s^2 + \epsilon^2\,[\vec{x}_\epsilon\cdot\vec{x}_\epsilon + 2s\,\vec{x}_{\epsilon\epsilon}\cdot\hat{k}] + 0(\epsilon^3) \qquad (32)$$

Therefore, the mean square value can be written as

$$< [\vec{x}(s,\epsilon)]^2 > = s^2 + \epsilon^2 < x_\epsilon^2 > + 2\epsilon^2 s\hat{k}\cdot< \vec{x}_{\epsilon\epsilon} > + 0(\epsilon^3) \qquad (33)$$

$< \vec{x}_{\epsilon\epsilon} >$ is given by (30) and $< x_\epsilon^2 >$ can be computed from eq. (16) as the following:

$$< x_\epsilon^2 > = < \int_0^s \left[\int_{t_1}^s \frac{d\tau}{n_o(\tau)}\right] \nabla_{T_1}\mu(\hat{k}t_1)dt_1 \cdot \int_0^s\int_{t_2}^s \frac{d\tau}{n_o(\tau)}\,\nabla_{T_2}\mu(\hat{k}t_2)dt_2 >$$

$$= \int_0^s\int_0^s \left[\int_{t_1}^s \frac{d\tau}{n_o(\tau)}\right]\left[\int_{t_2}^s \frac{d\tau}{n_o(\tau)}\right] < \nabla_{T_1}\mu(\hat{k}t_1)\nabla_{T_2}\mu(\hat{k}t_2) > dt_1 dt_2 \qquad (34)$$

Let

$$r = t_1 - t_2,\quad r_o = (1/2)\,(t_1 + t_2) \qquad (35)$$

then

$$< x_\epsilon^2 > = -2 < \mu^2 > \int_{-s}^{0} \frac{1}{r} N_r(r)dr \int_{-(r/2)}^{s+(r/2)} \left[\int_{r_o+(r/2)}^{s} \frac{d\tau}{n_o(\tau)} \int_{r_o-(r/2)}^{s} \frac{d\tau}{n_o(\tau)} \right] dr_o$$

$$-2 < \mu^2 > \int_{0}^{s} \frac{1}{r} N_r(r)dr \int_{r/2}^{s-(r/2)} \left[\int_{r_o+(r/2)}^{s} \frac{d\tau}{n_o(\tau)} \int_{r_o-(r/2)}^{s} \frac{d\tau}{n_o(\tau)} \right] dr_o \tag{36}$$

Combining eqs. (30), (33) and (36), we obtain

$$< [\vec{x}(s,\epsilon)]^2 > = s^2 + 2 < \mu^2 > \epsilon^2 \Bigg\{ 2 \int_0^s \frac{Nr(r)}{r} dr \int_r^s \left[\int_{r_2}^{s} \frac{d\tau}{n_o^2(\tau)} \right] dt_2$$

$$- \int_{-s}^{0} \frac{1}{r} N_r(r)dr \int_{-(r/2)}^{s+(r/2)} \left[\int_{r_o+(r/2)}^{s} \frac{d\tau}{n_o(\tau)} \int_{r_o-(r/2)}^{s} \frac{d\tau}{n_o(\tau)} \right] dr_o$$

$$- \int_{0}^{s} \frac{1}{r} N_r(r)dr \int_{r/2}^{s-(r/2)} \left[\int_{r_o+(r/2)}^{s} \frac{d\tau}{n_o(\tau)} \int_{r_o-(r/2)}^{s} \frac{d\tau}{n_o(\tau)} \right] dr_o \Bigg\}$$

$$+ 0(\epsilon^3) \tag{37}$$

Eq. (37) gives the mean square distance between the end points of a ray of length s. From this, the mean square value of the transverse displacement of the ray $\rho(s,\epsilon)$ can also be derived. Since $\rho(s,\epsilon)$ denotes the transverse displacement of the endpoint of a ray of length s from its original direction, then to the order of ϵ^2,

$$< \rho^2 > \; = < x_\epsilon^2 > \tag{38}$$

This is due to the fact $\vec{x}_\epsilon$ is transverse to the original ray direction. This quantity gives the mean square fluctuation of the transverse position of the beam spot from its original position and is an important quantity in designing optical receiving sensors.

Another mean square quantity of interest is the mean square deviations of the direction of the ray of length s,

$$< \left(\frac{d\vec{x}}{ds} - \hat{k}\right)^2 > = \epsilon^2 < \left(\frac{d\vec{x}_\epsilon}{ds}\right)^2 > + 0(\epsilon^3) \tag{39}$$

$$< \left(\frac{d\vec{x}_\epsilon}{ds}\right)^2 > = < \int_0^s \frac{1}{n_o(s)} \nabla_{T_1} \mu(\hat{k}t_1)dt_1 \cdot \int_0^s \frac{1}{n_o(s)} \nabla_{T_2} \mu(\hat{k}t_2)dt_2 >$$

$$= \frac{1}{n_o^2(s)} \int_0^s dt_1 \int_0^s dt_2 \nabla_{T_1} \cdot \nabla_{T_2} < \mu(\hat{k}t_1) \mu(\hat{k}t_2) >$$

$$= \frac{4 < \mu^2 >}{n_o^2(s)} [N(s) - 1 - s \int_0^s \frac{1}{r} N_r dr] \tag{40}$$

If $\theta(s,\epsilon)$ is the angle between $\frac{d\vec{x}}{ds}$ (s,ϵ) and the original ray direction $\hat{k}$, then

$$< \left(\frac{d\vec{x}}{ds} - \hat{k}\right)^2 > = 2 (1 - < \cos \theta >)$$

$$= \frac{4\epsilon^2 < \mu^2 >}{n_o^2(s)} [N(s) - 1 - s \int_0^s \frac{1}{r} N_r dr] + 0(\epsilon^3) \tag{41}$$

For s much greater than the correlation length of $N(r)$, and for small deviation of the ray direction (41) reduces to

$$< \theta^2 > \approx 4Ds \tag{42}$$

where

$$D(s) = - \frac{\epsilon^2 < \mu^2 >}{n_o^2(s)} \int_0^\infty \frac{Nr(r)}{r} dr \qquad (43)$$

is the ray diffusion coefficient.

We note here that all the expressions we have derived will reduce to the corresponding expressions derived by Keller (1962) in the case $n_o(z) \to 1$.

4. <u>An Example</u>

Let the refractive index for the background medium be

$$n_o(z) = e^{-\alpha z} \qquad z > 0 \qquad (44)$$

We note that $\alpha = 0$ corresponds to the free space while negative values of α correspond to propagation into "optically denser" media. A ray is assumed to start from the origin in the $\hat{k}$ direction. Because of the small random variation of the refractive index, the direction and the position of the endpoint of the ray will fluctuate about their mean values. After the ray has traveled a distance s along the ray path, the mean square fluctuation of the position of the end points of the ray is given by $< x^2(s,\epsilon) > - s^2$ and the mean square fluctuation of the transverse displacement is $< \rho^2 >$. Equations (37), (38) can be used for the computation. For explicite results, we have assumed a normalized correlation function for $\mu(\vec{x})$

$$N(r) = e^{-(r^2/a^2)} \tag{45}$$

After some elementary integrations, we obtain for s large compared to the correlation length a

$$< \rho^2(s,\epsilon) > = < x_\epsilon^2(s,\epsilon) >$$

$$= \frac{2\epsilon^2 < \mu^2 > \sqrt{\pi}}{a\alpha^2} \; e^{2\alpha s} \Bigg\{ 2s - \frac{2}{\alpha}\,(1 - e^{-\alpha s}) - \frac{2a}{\sqrt{\pi}}$$

$$- \frac{1}{\alpha}\, e^{(\alpha^2 a^2/4)} \, [1 - \mathrm{erf}(\alpha a/2)] + \frac{1}{\alpha}\, e^{(\alpha^2 a^2/4)} (2e^{-\alpha s} - e^{-2\alpha s}).$$

$$\cdot [1 + \mathrm{erf}(\alpha a/2)] \Bigg\} \; + \; 0(\epsilon^3) \tag{46}$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \tag{47}$$

and

$$< x^2(s,\epsilon) > - s^2 = < \rho^2(s,\epsilon) > - \frac{2\epsilon^2 < \mu^2 > \sqrt{\pi}}{a\alpha^2} \, e^{2\alpha s} \Bigg\{ \alpha s^2 - \frac{s}{2}$$

$$- \frac{\alpha a s}{\sqrt{\pi}} + \frac{s}{2}\, e^{a^2\alpha^2 - 2\alpha s} [1 + \mathrm{erf}(\alpha a)] \Bigg\} \; + 0(\epsilon^3) \tag{48}$$

For $\alpha \to 0$, (46) and (48) reduce respectively to

$$< \rho^2(s,\epsilon) > = \frac{2\epsilon^2 < \mu^2 >}{3a} \, (a^3 - 3as^2 + 2\sqrt{\pi}\, s^3) + 0(\epsilon^3) \tag{49}$$

$$< x^2(s,\epsilon) > - s^2 = \frac{2\epsilon^2 < \mu^2 >}{3a} \left(a^3 + 3as^2 - \frac{3\sqrt{\pi}}{2} a^2 s - \sqrt{\pi}\, s^3\right) + 0(\epsilon^3) \tag{50}$$

which correspond to the case of homogeneous background.

Equations (46) and (48) are plotted in Figures 1 and 2 for different values of α for $s > 5a$. For a fixed amount of random variation $\epsilon\mu(\vec{x})$ in the refractive index, the fluctuations for cases of positive values of α are larger than that for the case when $\alpha = 0$. The greater value of α, the larger the fluctuations. On the other hand, for negative values of α, the fluctuations are smaller than that for the case when $\alpha = 0$, and the greater value of $|\alpha|$, the smaller the fluctuations. Therefore, if a light beam is traveling in a slightly random medium, for fixed amount of random fluctuation of the refractive index, the beam will "diffuse" slower if the propagation is in a direction that the background is getting "optically denser." This result can be attributed to the fact that the percentage fluctuation of refractive index the ray sees is smaller in an optically denser medium, hence less fluctuations in ray direction and displacement.

It is of interest to see what indeed are the effects of the stratified background on the ray statistics by normalizing $< \rho^2(s,\epsilon) >$ with respect to the local percentage fluctuation of the refractive index. Define

$$\text{Normalized} < \rho^2(s,\epsilon) > = < \rho^2(s,\epsilon) > / [\epsilon^2 < \mu^2 > \sqrt{\pi}\, a^2/n_o^2(s)] \tag{51}$$

This quantity gives the mean square fluctuation of the transverse displacement of the ray at s for a fixed amount of local percentage fluctuation of refractive index. Eq. (51) is plotted in Fig. 3 for different values of α. For this particular example, the fluctuations are greater for negative values of α and smaller for positive values of α. For other forms of stratification of the background medium, the result may not be the same.

We also note that for all cases, the fluctuations increase as s increases. This is physically expected.

Similar results can be obtained for $< \theta^2 >$.

5. Conclusion

The statistics of a light ray propagating in a slightly random medium with stratified background is studied. We concentrate on the case of normal incidence. Expressions for mean square fluctuations of ray displacement, transverse displacement and ray direction are derived to the order of ϵ^2 where ϵ is the small parameter which measures the random fluctuation of the medium. Those expressions are shown to be reducible to the expressions Keller (1962) obtained when the background becomes homogeneous. An example is considered. It is found that for a fixed amount of random fluctuation in the refractive index, fluctuations of the ray are smaller for propagation into optically denser background media ($|n_o| > 1$) than for background media with $|n_o| < 1$. Therefore, a ray traveling in a slightly random medium will "diffuse" slower if the background is getting optically denser, as is expected physically.

Our derivation is valid for s small compared to the "diffusion length" 1/D of the medium. Another approach is to derive a

Fokker-Planck equation for the probability density of finding the ray at a given position with a given direction for a given path length s. The solution of this Fokker-Planck equation determines the statistics of the ray. This approach is valid for s much greater than the correlation distance of the random medium. For weakly random media so that the diffusion length is much greater than the correlation length, there may be overlapping regions where both methods are applicable.

We also note that the assumption of isotropic random variation of the refractive index in our derivation can be removed with the consequence of obtaining more complex expressions for the average quantities.

REFERENCES

Bremmer, H., "Random Volume Scattering," Radio Science, 68D, 967-981 (Sept., 1964).

Chernov, L. A., "Wave Propagation in a Random Medium," McGraw-Hill Book Co., N. Y. (1960).

Keller, J. B., "Wave Propagation in Random Media," Proc. Sym. in Appl. Math., V. 13, p. 227 (1962).

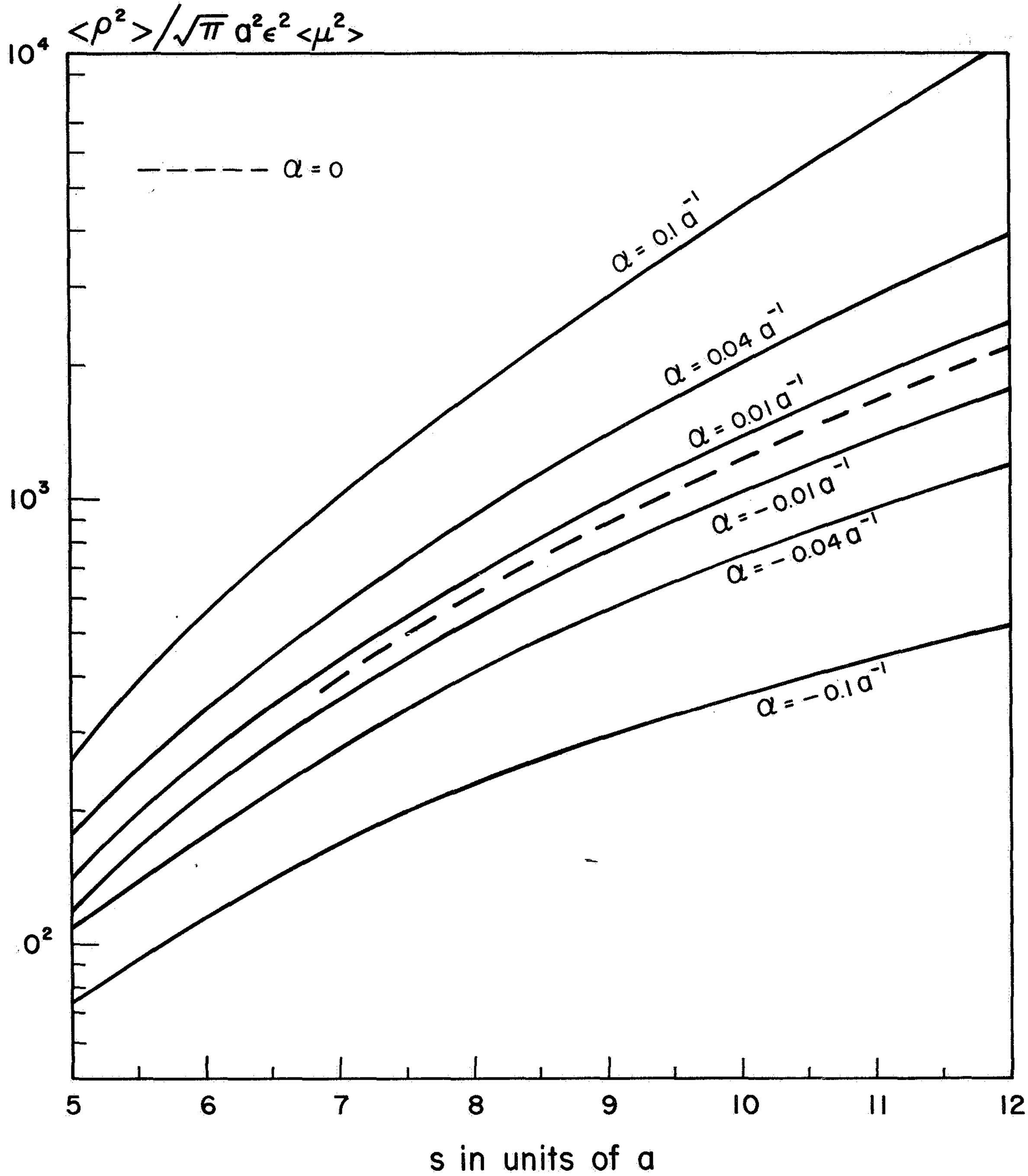

Fig. 1 Mean Square Transverse Displacement of the Ray as a Function of Path Length

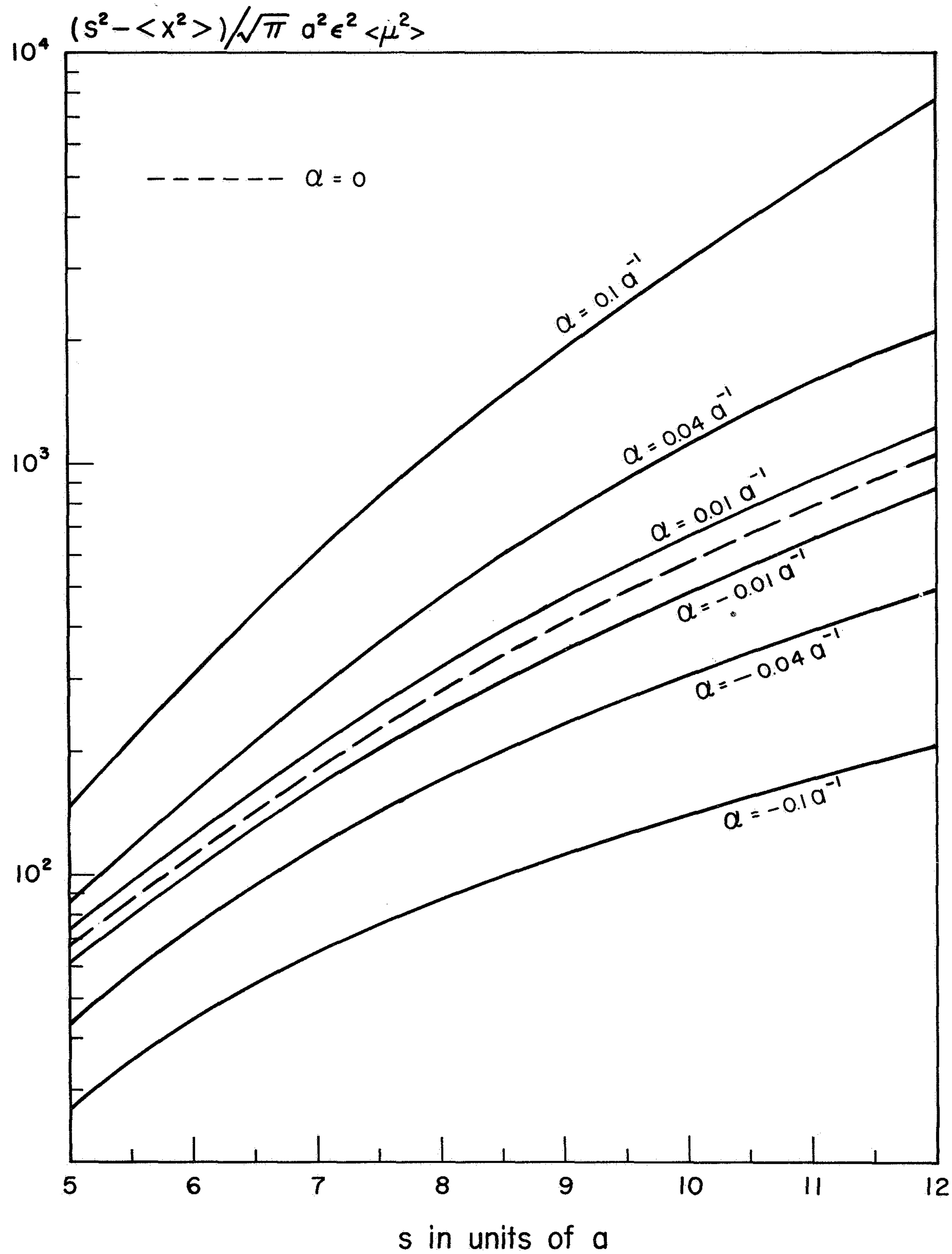

Fig. 2 Mean Square Deviation of the Displacement of the Ray as a Function of Path Length

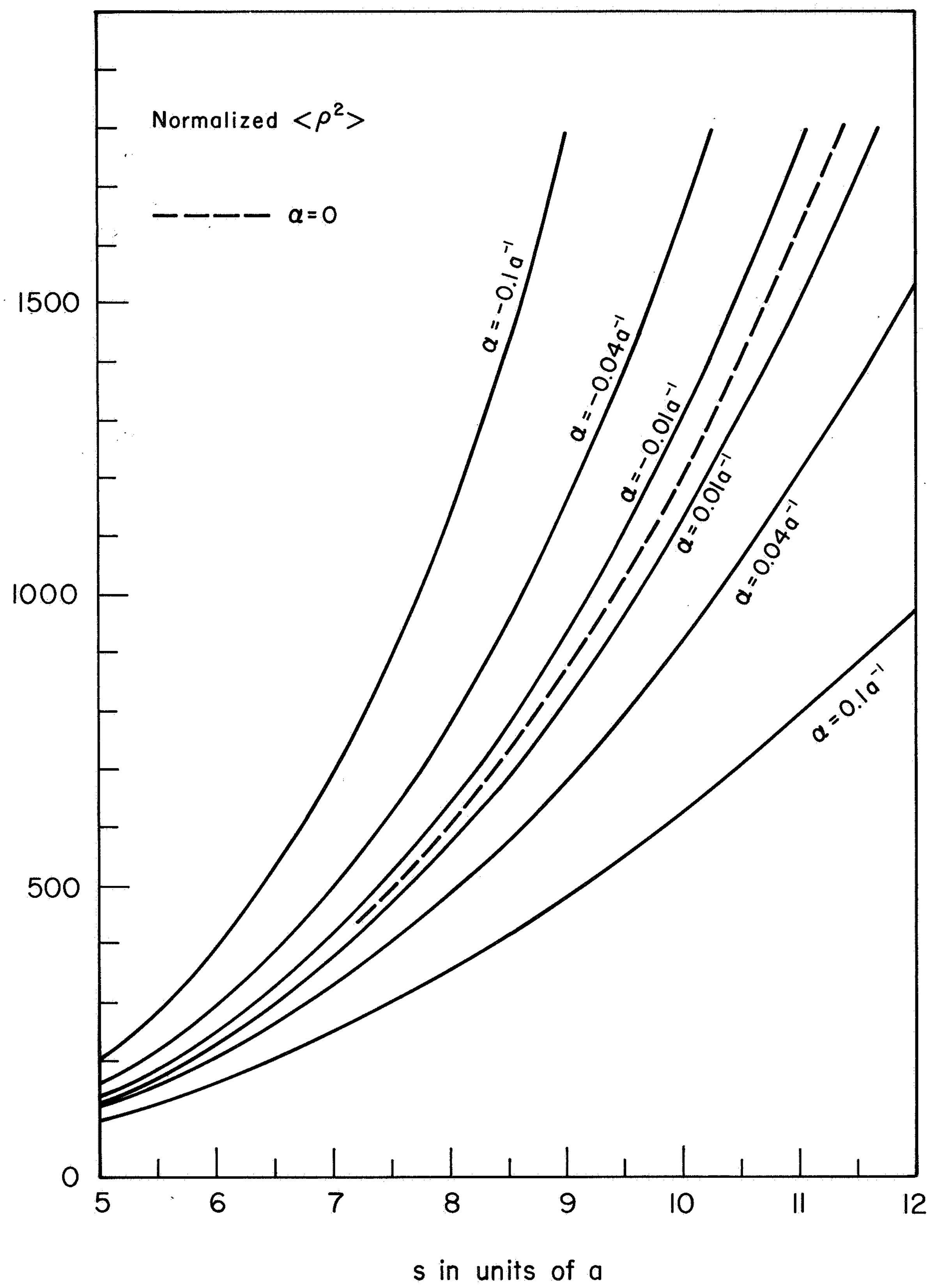

Fig. 3 Normalized Mean Square Transverse Displacement as a Function of Path Length

DIFFUSION OF RAYS IN RANDOM MEDIA

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Abstract

This paper is concerned with propagation of rays in a random medium. For ray path larger than the correlation distance the probability density of finding the ray satisfies a Fokker-Planck equation. It is shown that the coefficient of the equation is related to the correlation function of the logarithm of the refractive index.

The Fokker-Planck equation is treated for three "initial" conditions. Technique for handling the general "initial" condition has been also indicated.

1. Introduction

The Fokker-Planck equation has been used to treat a large class of physical problems. These problems include studies of, for example, Brownian motion (Chandrasekhar, 1943), plasma physics (Montgomery and Tidman, 1964), particle scattering (Salpeter and Treiman, 1964), radiative transfer (Chandrasekhar, 1960), noise theory (Stratonovich, 1963), propagation of sound (Lyon, 1959), stellar mechanics (Chandrasekhar, 1943), etc. Recently the equation has been applied to problems of ray propagation in a random medium (Chernov, 1960; Keller, 1962; Bremmer, 1964; Komisarov, 1966). Chernov and Keller were concerned with the homogeneous case in which the probability density function is independent of coordinates while Bremmer and Komisarov were concerned with the forward scattering approximation. Both of these approaches require the fluctuations to be weak and they become invalid when fluctuations in dielectric constant are strong.

The starting point of the present treatment is the well-known ray equation. The form of the equation suggests that the random medium be specified by the statistical properties of $\mu(\vec{x}) = \log n(\vec{x})$ where $n(\vec{x})$ is the refractive index. In a weakly random medium μ is proportional to the fluctuating part of the refractive index. However, in general it is more natural to specify μ. In this paper, we shall assume the function μ to be a homogeneous random field with a correlation function $B(\vec{x})$ defined by

$$B(\vec{x}_1-\vec{x}_2) = < \mu(\vec{x}_1)\mu(\vec{x}_2) > \tag{1}$$

where $< >$ denotes an ensemble average. The random field is isotropic if the correlation function depends only on $|\vec{x}| = r$. We shall be concerned here exclusively with the isotropic case in which the diffusion coefficient defined by

$$D = -\int_0^\infty \frac{B'(r)}{r}\, dr \tag{2}$$

plays an important role. In (2) as well as elsewhere in this paper a prime is used to denote differentiation with respect to the argument. When defined this way D is related to the correlation distance of the process μ. We list in the following two examples:

Example 1. If $B(\vec{x}) = <\mu^2(\vec{x})> \exp - r^2/\ell^2$

then $D = <\mu^2(\vec{x})> \sqrt{\pi}/\ell$

Example 2. If $B(\vec{x}) = <\mu^2(\vec{x})> [1 + (r/\ell)^2]^{-2}$

then $D = 3 <\mu^2(\vec{x})> \pi/4\ell$

In both of these examples ℓ is the correlation distance. If the medium is weakly random 1/D is much larger than ℓ, but it may approach ℓ when fluctuations in μ become strong.

2. The Fokker-Planck Equation

The position $\vec{x}$ of a ray whose path length is s from a given initial position is determined by the well-known ray equation. For our purposes we write

$$\begin{aligned} \vec{x}'(s) &= \vec{v} \\ \vec{v}'(s) &= \vec{F}(\vec{x},\vec{v}) \end{aligned} \tag{3}$$

The vector $\vec{v}$ is a unit vector which specifies the direction of the ray at $\vec{x}$. The vector $\vec{F}$ is given by

$$\vec{F} = \nabla\mu(\vec{x}) - \vec{v}\,\vec{v}\cdot\nabla\mu(\vec{x}) \tag{4}$$

and it has a vanishing mean. The "force" $\vec{F}$ is random since μ is random. The set of equations (3) is similar to one that describes the Brownian motion. Therefore, there is associated a Fokker-Planck equation. For a path much larger than the correlation distance the probability $\omega(\vec{x},\vec{v},s)$ of finding the ray at $\vec{x}$ with the ray direction $\vec{v}$ after propagating a path length s from its initial position satisfies the equation (Stratonovich, 1963; So, 1967)

$$\begin{aligned}(\partial/\partial s + \vec{v}\cdot\nabla)\omega(\vec{x},\vec{v},s) &= \frac{\partial}{\partial v_i} \int_{-\infty}^{0} d\sigma < F_j(\vec{x}+\sigma\vec{v},\vec{v}) \frac{\partial F_i}{\partial x_j} > \sigma\omega \\ &- \frac{\partial}{\partial v_i} \int_{-\infty}^{0} d\sigma < F_j(\vec{x}+\sigma\vec{v},\vec{v}) \frac{\partial F_i}{\partial v_j} > \omega \\ &+ \frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j} \int_{-\infty}^{0} d\sigma < F_j(\vec{x}+\sigma\vec{v},\vec{v})\, F_i > \omega \end{aligned} \tag{5}$$

Terms involving repeated indices are to be summed from 1 to 3. Eq. (5) is quite general and it applies as long as $\vec{F}$ is not explicitly dependent on s. For the special form of $\vec{F}$ given by (4) in which we additionally assume μ to be a homogeneous and isotropic random field, (5) reduces to

$$(\partial/\partial s + \vec{v}\cdot\nabla)\omega(\vec{x},\vec{v},s) = D \frac{\partial}{\partial v_i} (\delta_{ij}-v_i v_j) \frac{\partial\omega}{\partial v_j} \tag{6}$$

The right hand side of (6) can be identified as the Laplacian expression on a unit sphere in $\vec{v}$-space. If we let θ and ϕ be the polar angle and the azimuthal angle of $\vec{v}$ respectively and let $c = \cos\theta$, then (4) can also be written as

$$(\partial/\partial s + \vec{v}\cdot\nabla)\omega(\vec{x},\vec{v},s) = D\left[\frac{\partial}{\partial c}(1-c^2)\frac{\partial\omega}{\partial c} + \frac{1}{1-c^2}\frac{\partial^2\omega}{\partial\phi^2}\right] \tag{7}$$

Eq. (7) is a diffusion equation and is the equation of interest. We wish to consider it for an unbounded random medium with three "initial" conditions. They are

1) "Initial" condition $\omega(\vec{x},\vec{v},o) = \delta(\theta)/2\pi$ (8)

2) "Initial" condition $\omega(\vec{x},\vec{v},o) = \delta(\theta)\delta(\vec{x})/2\pi$ (9)

3) "Initial" condition $\omega(\vec{x},\vec{v},o) = \delta(\vec{x})/4\pi$ (10)

We consider each of these in the following.

3. Examples

(1) Initial Condition $\omega(\vec{x},\vec{v},o) = \delta(\theta)/2\pi$

Let sources of rays be uniformly distributed and when emitted all rays be directed along the z-axis. In this case ω depends only on θ and s. The Fokker-Planck equation (7) reduces to

$$\partial\omega/\partial s = D\frac{\partial}{\partial c}(1-c^2)\frac{\partial\omega}{\partial c} \tag{11}$$

with the initial condition given by (8). This problem has been considered by Chernov (1960) and Keller (1962). The solution is given by

$$\omega(c,s) = \frac{1}{4\pi}\sum_{n=o}^{\infty}(2n+1)P_n(c)e^{-n(n+1)Ds} \tag{12}$$

where $P_n(c)$ is the Legendre polynomial of order n. The expression given by (12) is already convenient for large values of Ds. In particular, $\omega(c,\infty) = 1/4\pi$, indicating that rays are uniformly distributed in all directions for large s.

For small values of Ds, in order to obtain a more rapidly convergent expression which is valid for arbitrary θ, the following procedure is applied. For $\theta < \pi$, the Legendre polynomial has the integral representation of Laplace and Mehler.

$$P_n(c) = \frac{\sqrt{2}}{\pi} \int_\theta^\pi d\alpha \frac{\sin[(n+\frac{1}{2})\alpha]}{(\cos\theta-\cos\alpha)^{1/2}} \tag{13}$$

Eq. (12) then becomes

$$\omega(c,s) = \frac{\sqrt{2}}{4\pi^2} \int_\theta^\pi d\alpha \frac{G(\alpha,s)}{(\cos\theta-\cos\alpha)^{1/2}} \tag{14}$$

where

$$G(\alpha,s) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (2n+1)\sin[(n+\frac{1}{2})\alpha]e^{-n(n+1)Ds} \tag{15}$$

$$= \frac{1}{2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{+\infty} dxe^{-2\pi imx}(2x+1)\sin[(x+\frac{1}{2})\alpha]e^{-x(x+1)Ds} \tag{16}$$

The Poisson sum formula (Morse and Feshback, 1953) has been used to obtain (16) from (15). For $Ds \ll 1$, only the m = 0 term is important in (16) and we obtain in (14)

$$\omega(c,s) \approx \frac{1}{4\sqrt{2}\,(\pi Ds)^{3/2}} \int_\theta^\pi d\alpha \frac{\alpha e^{-\alpha^2/4Ds}}{(\cos\theta-\cos\alpha)^{1/2}}, \quad Ds \ll 1 \tag{17}$$

The integral can be computed in the limit $Ds \to 0$ by expanding the integrand about $\alpha = \theta$. The final expression is

$$\omega(c,s) \approx \frac{1}{4\pi Ds} \left(\frac{\theta}{\sin\theta}\right)^{1/2} e^{-\theta^2/4Ds} \quad , \; Ds \ll 1 \tag{18}$$

which is valid for arbitrary angle θ.

The expression (12) can also be used to compute certain average quantities. These have been given in the literature (Chernov, 1960). For example the mean value of the cosine of the ray direction is

$$< c > = \exp - 2Ds \tag{19}$$

The mean displacement and the mean square displacement of the ray of length s are given by

$$\begin{aligned} < x > &= < y > = 0 \\ < z > &= (1 - \exp - 2Ds)/2D \\ < \rho^2 > = < x^2 + y^2 > &= \frac{2s}{3D} - \frac{1}{D^2}\left(\frac{4}{9} - \frac{e^{-2Ds}}{2} + \frac{e^{-6Ds}}{18}\right) \\ < z^2 > &= s/3D - (1-e^{-6Ds})/18D^2 \end{aligned} \tag{20}$$

In the limit of small Ds, we have

$$< z > = s, \quad < \rho^2 > = 4Ds^3/3, \quad < z^2 > = s^2 - 2Ds^3 \tag{21}$$

These averages are listed here for the sake of completeness and for the ease of comparison with the following examples.

(2) Initial Condition $\omega(\vec{x},\vec{v},o) = \delta(\theta)\delta(\vec{x})/2\pi$

The example considered in (1) is rather unrealistic since it requires sources of the ray to be uniformly distributed in $\vec{x}$-space. If there is just a single source we then have the present problem. In practice this requires a highly directional source of the ray such as a laser beam. The equation of interest is given by (7) whose analytic solution does not seem to be attainable. We shall follow Salpeter and Treiman (1964) and work with moments.

The zeroth spatial moment is defined by

$$\omega^{(o)}(c,s) = \int \omega(\vec{x},\vec{v},s)\ d^3x \tag{22}$$

which is the probability density of finding a ray of path length s from the origin and with a polar angle of the ray $\cos^{-1} c$. By taking the zeroth moment of (7) we find that $\omega^{(o)}$ satisfies (11) with the initial condition (8). Therefore, $\omega^{(o)}$ is identical to that given by (12) and this is a reasonable expectation on physical grounds.

For higher moments we first consider the moments in the coordinate along the initial ray direction. Define the n th moment by

$$\omega_z^{(n)}(\vec{v},s) = \int z^n \omega(\vec{x},\vec{v},s)\ d^3x, \qquad n = 1,2,\ \ldots \tag{23}$$

Multiplying (7) by z^n and integrating over $\vec{x}$, we note that $\omega_z^{(n)}$ is not a function ϕ and hence must satisfy the equation

$$\frac{\partial\omega_z^{(n)}}{\partial s} - D\frac{\partial}{\partial c}(1 - c^2)\frac{\partial\omega_z^{(n)}}{\partial c} = cn\omega_z^{(n-1)} \qquad n = 1,2,\ldots \tag{24}$$

with the initial condition $\omega_z^{(n)}(c,o) = 0$. The solution of (24) for $\omega_z^{(n)}$ expressed in terms of $\omega_z^{(n-1)}$ can be obtained by the standard method. We have

$$\omega_z^{(n)}(c,s) = \frac{n}{4\pi}\int ds' \sum_m (2m+1)P_m(c)e^{-m(m+1)D(s-s')}$$

$$\cdot \int d\Omega' P_m(c')c'\omega_z^{(n-1)}(c',s') \quad n = 1,2,\ldots \tag{25}$$

where $d\Omega'$ is the solid angle element in $\vec{v}$-space. Since $\omega_z^{(o)}$ has already been found to be given by (12), in principle, (25) enables us to compute all orders of moments. We shall compute in the following first and second moments.

The first moment is obtained by substituting (12) for $\omega^{(o)}$ in the integrand of (25) and carry out the integration. The final expression is

$$\omega_z^{(1)}(c,s) = \frac{1}{8\pi D}\sum_{m=o}^{\infty} P_m(c)\left\{e^{-m(m-1)Ds} - e^{-(m+1)(m+2)Ds}\right\} \tag{26}$$

Again, (26) is already in a convenient form for large values of Ds. For small values of Ds, the same procedure as that used in example 1 can be applied to (26). We obtain

$$\omega_z^{(1)}(c,s) \approx \frac{1}{4\pi D}\left(\frac{\sin\theta}{\theta}\right)^{1/2} e^{-\theta^2/4Ds}, \quad Ds \ll 1 \tag{27}$$

When (26) is averaged over all the angles, it gives the average z-position of the ray of length s irrespective of the direction of the ray,

$$< z(s) > \;=\; (1-e^{-2Ds})/2D \tag{28}$$

We note that (28) is identical to the corresponding expression given by (20) for example 1.

Another quantity of some interest is

$$\bar{z}(c,s) = \omega_z^{(1)}(c,s)/\omega^{(o)}(c,s) \tag{29}$$

This is the average z-position of the ray with the ray direction $\cos^{-1}c$ and path length s. For $Ds \ll 1$,

$$\bar{z}(c,s) \approx Ds(\sin\theta/\theta), \quad Ds \ll 1 \tag{30}$$

From (30) we can see that the "forward scattering" approximation is good for $Ds \ll 1$ and is fulfilled only when the medium is weakly random. Eq. (30) also shows how the average z-position depends on the direction of the ray.

Higher moments of z can be obtained similarly from (25) in succession. For example

$$\omega_z^{(2)}(c,s) = \frac{1}{4\pi D^2} \sum_{m=o}^{\infty} P_m(c) e^{-m(m+1)Ds} \left\{ \left(\frac{m+1}{2m+3} - \frac{m}{2m-1} \right) Ds \right.$$

$$\left. - \frac{m+1}{2(2m+3)^2} [1-e^{-2(2m+3)Ds}] - \frac{m}{2(2m-1)^2} [1-e^{2(2m-1)Ds}] \right\} \tag{31}$$

Averaging (31) with respect to all angles, we have

$$< z^2(s) > = [6Ds-(1-e^{-6Ds})]/18D^2 \tag{32}$$

This is the mean square z-position of the ray of length s and is seen identical to the corresponding expression in (20) for example 1. Higher moments can be considered in the same manner. Since they are not of practical interest we shall now turn our attention to spatial moments transverse to the original direction of the ray.

Define the n th moment by

$$\omega_x^{(n)}(c,\phi,s) = \int x^n \omega(\vec{x},\vec{v},s)\, d^3x \tag{33}$$

Multiply the diffusion equation (7) by x^n and integrate over spatial coordinates; we obtain

$$\frac{\partial \omega_x^{(n)}}{\partial s} - D\left\{\frac{\partial}{\partial c}(1-c^2)\frac{\partial \omega_x^{(n)}}{\partial c} + \frac{1}{1-c^2}\frac{\partial^2 \omega_x^{(n)}}{\partial \phi^2}\right\} = n(1-c^2)^{1/2}\cos\phi\, \omega_x^{(n-1)} \qquad n = 1,2,3,\ldots \tag{34}$$

with the "initial" condition $\omega_x^{(n)}(c,\phi,o) = 0$.

Eq. (34) has the solution in the form

$$\omega_x^{(n)}(c,\phi,s) = n\int_0^s ds' \sum_{\ell=o}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(c,\phi)e^{-\ell(\ell+1)D(s-s')}$$

$$\cdot \int d\Omega' \, Y_\ell^m(c',\phi')\sqrt{1-c'^2}\cos\phi'\, \omega_x^{(n-1)}(c',\phi',s') \tag{35}$$

where Y_m^ℓ is the spherical harmonic function. The first moment is obtained by substituting $\omega^{(o)}$ given by (12) in the integrand of (35). Using the orthogonality relation of the spherical harmonic functions, the integation yields

$$\omega_x^{(1)}(c,\phi,s) = \frac{(1-c^2)^{1/2}\cos\phi}{8\pi D}\sum_{n=1}^{\infty}\frac{1}{n}\,[P_n'(c)-P_{n-1}'(c)]$$

$$\cdot\,[e^{-n(n-1)Ds}-e^{-n(n+1)Ds}] \tag{36}$$

Following the same procedure as shown in example 1 the asymptotic expression for $Ds \ll 1$ can be shown to be

$$\omega_x^{(1)}(c,\phi,s) \sim \frac{\cos\phi}{4\pi D}\left(\frac{\theta}{\sin\theta}\right)^{1/2}\left(\frac{1-\cos\theta}{\theta}\right)e^{-\theta^2/4Ds} \tag{37}$$

The average x-position of the ray can be obtained by integrate (36) over all angles. The integration is trivial and gives the physically expected result

$$< x(s) > \; = 0 \tag{38}$$

Higher moments in x can be obtained similarly in succession from (36). Among them, $\omega_x^{(2)}(c,\phi,s)$ is of particular interest because it gives the mean square spread of the ray of length s in the x-direction. The general expression for $\omega_x^{(2)}$ is rather lengthy and will be given in the Appendix. The mean square value of the x-position of the ray is obtained by integrate $\omega_x^{(2)}$ over the solid angle. The result is

$$< x^2(s) > \; = \frac{s}{3D} - \frac{1}{D^2}\left(\frac{2}{9} - \frac{e^{-2Ds}}{2} + \frac{e^{-6Ds}}{36}\right) \tag{39}$$

Due to axial symmetry of the problem an identical expression for $< y^2(s) >$ can be obtained in a similar manner.

We note that the mean and the mean square position of the ray for the present problem are found to be identical to the mean and the mean square displacement of the ray respectively in example 1. This is to be expected since the medium is assumed to be statistically homogeneous and hence the problem possesses translational symmetry. Of course this is no longer true if we allow the background medium to be, for example, inhomogeneous.

(3) Initial Condition $\omega(\vec{x},\vec{v},o) = \delta(\vec{x})/4\pi$

The source of the ray is at the origin emitting rays with equal probability in all directions. This problem can be treated in the similar manner as in example 2. The zeroth moment is obtained by solving (11) with the given initial condition. We have

$$\omega^{(o)}(c,s) = 1/4\pi \tag{40}$$

which is a constant. This is to be expected because of the isotropic property of the source and the medium. The first moments can be computed by substituting (40) for $\omega^{(o)}$ in (25) and (35). Higher moments are then obtained in succession. We list these moments in the following:

$$\omega_x^{(1)}(c,\phi,s) = (1-c^2)^{1/2}\cos\phi(1-e^{-2Ds})/8\pi D \tag{41}$$

$$\omega_y^{(1)}(c,\phi,s) = (1-c^2)^{1/2}\sin\phi(1-e^{-2Ds})/8\pi D \tag{42}$$

$$\omega_z^{(1)}(c,s) = c(1-e^{-2Ds})/8\pi D \tag{43}$$

$$< x(s) > = < y(s) > = < z(s) > = 0 \tag{44}$$

$$\omega_x^{(2)}(c,\phi,s) = \frac{1}{4\pi D^2}\left\{\frac{1}{3}[Ds - \frac{1}{2}(1-e^{-2Ds})] + \frac{1}{2}\left[\frac{1}{3} - c^2 - (1-c^2)\mathrm{coo}2\phi\right]\right.$$

$$\left.\cdot\left[\frac{1}{6}(1 - e^{-6Ds}) - \frac{1}{4}(1-e^{-4Ds})e^{-2Ds}\right]\right\} \tag{45}$$

$$\omega_z^{(2)}(c,\phi,s) = \frac{1}{4\pi D^2}\left\{\frac{1}{3}[Ds - \frac{1}{2}(1 - e^{-2Ds})]\right.$$

$$\left.+\left(c^2 - \frac{1}{3}\right)\left[\frac{1}{6}(1-e^{-6Ds}) - \frac{1}{4}(1-e^{-4Ds})e^{-2Ds}\right]\right\} \tag{46}$$

$$< x^2(s) > = < y^2(s) > = < z^2(s) > = [Ds - \frac{1}{2}(1-e^{-2Ds})]/3D^2 \tag{47}$$

$$< r^2(s) > = [Ds - \frac{1}{2}(1 - e^{-2Ds})]/D^2 \tag{48}$$

Another quantity of interest is the average position of the ray of path length s with a given direction (θ,ϕ). From (41), (42) and (43), we have

$$\omega_x^{(1)}\hat{x} + \omega_y^{(1)}\hat{y} + \omega_z^{(1)}\hat{z})/\omega^o = \hat{r}(1 - e^{-2Ds})/2D \tag{49}$$

where $\hat{x}$, $\hat{y}$, $\hat{z}$ and $\hat{r}$ are unit vectors. For $Ds \ll 1$, the position of the end point of the ray in the direction of $\hat{r}$ with path length s is $s\hat{r}$, just as if it is travelling in a regular homogeneous medium. For larger values of Ds so that the multiple scattering effect on the ray becomes important, the average position of the ray is still in the original direction $\hat{r}$, but the distance from the origin, the starting point of the ray, is always less than s.

4. Discussion

In the preceding section we have discussed three examples. These examples are highly idealized with very special "initial" conditions. In general a radiator has a radiation pattern which is not necessarily infinitely sharp nor isotropic. It has a radiation pattern $f(\theta,\phi)$ which is a function of both angles. We wish to indicate in the following steps to take to treat this general case.

Let us first consider the solution of the diffusion equation (7) with the "initial" condition

$$\omega(\vec{x},\vec{v},o) = \delta(\theta-\theta_o)\delta(\phi-\phi_o)\delta(\vec{x})/2\pi \tag{50}$$

We note that the condition (50) is identical to (9) if we make a suitable coordinate transformation. This suggests that a solution to the problem with the initial condition (50) can be obtained through a similar coordinate transformation of the solution obtained in example 2. Let the angle between the radial lines of coordinates (θ_o,ϕ_o) and (θ,ϕ) be α. The desired solution to the present problem is obtained if we substitute $\cos\theta$ in example 2 by $\cos\alpha$. Here

$$\cos\alpha = \cos\theta_o\cos\theta + \sin\theta_o\sin\theta\cos(\phi - \phi_o) \tag{51}$$

For example, the zeroth moment is found from (12) as

$$\omega^{(0)}(\theta,\phi,s) = \frac{1}{4\pi}\left\{1 + \sum_{n=1}^{\infty} (2n+1)[P_n(\cos\theta_o)P_n(\cos\theta) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\right.$$

$$\left.\cdot\ P_n^m(\cos\theta_o)\ P_n^m(\cos\theta)\cos m(\phi-\phi_o)]\right\} e^{-n(n+1)Ds} \tag{52}$$

where the addition theorem has been used. Higher moments can be obtained similarly.

Now let us assume the radiator has a "probable" radiation pattern $f(\theta,\phi)$, i.e.

$$\omega(\vec{x},\vec{v},o) = f(\theta,\phi)\delta(\vec{x}) \tag{53}$$

The function f is assumed to have been normalized over the solid angle in order to be a probability density function. Expand f on the surface of unit sphere in terms of spherical harmonics

$$f(\theta,\phi) = \sum_{\ell=o}^{\infty}\sum_{p=\ell}^{\ell} A_{\ell p}\, Y_{\ell}^{p}(\theta,\phi) \tag{54}$$

The coefficients $A_{\ell p}$ can be computed simply since f is known. Evaluate (54) at (θ_o,ϕ_o), substitute it into (52) and integrate over the solid angle $d\Omega_o$; we obtain

$$\omega^{o}(\theta,\phi,s) = \sum_{n=o}^{\infty}\sum_{m=-n}^{n} A_{nm}\, Y_{n}^{m}(\theta,\phi)e^{-n(n+1)Ds} \tag{55}$$

Similar technique can be applied to obtain higher moments for the "initial" condition (53). (See Appendix)

5. Conclusion

The problem of propagation of light rays in a random medium is investigated here in a unified manner without the usual assumption that the medium be weakly random. Starting from the well-known ray equation, a generalized Fokker-Planck equation for the probability density of finding the ray in the medium is derived. The equation is general and can be applied to random media with anisotropic irregularities or with inhomogeneous background. The only assumption is that the random scatters have a correlation distance small compared to the path length of interest. Three problems corresponding to three different "initial" conditions of the ray are studied for a statistically homogeneous and isotropic medium. Exact expressions for the probability density and/or spatial moments are derived. The quantity D, the diffusion coefficient of the medium for the ray, is found to play an important role in all cases. It is found that in the limit of $Ds \ll 1$ and for cases that the total deflection of the ray due to random scattering is small, the so-called "forward scattering" approximation is applicable. This is likely to happen only for weakly random medium. For media with strong fluctuations in the refractive index, even for $Ds \ll 1$, the deflection in the ray direction may be large and the asymptotic expressions derived should be used. On the other hand, for the other extreme limit such that $Ds \gg 1$, the ray has suffered a large number of scattering in the medium, it is shown that the distribution becomes isotropic in all directions.

Most radiators have a definite radiation pattern. This corresponds to some arbitrary "initial" condition. Steps are shown for treating such a general case.

APPENDIX

1. From (35) and (36) we have

$$\omega_x^{(2)}(c,\phi,s) = 2\int_0^s ds' \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(c,\phi)e^{-\ell(\ell+1)D(s-s')}$$

$$\cdot \int d\Omega' Y_\ell^m(c',\phi')\frac{\cos^2\phi'}{8\pi D}(1-c'^2)\sum_{n=1}^{\infty}\frac{1}{n}$$

$$\cdot [P_n'(c') - P_{n-1}'(c')]\,[e^{-n(n-1)Ds'} - e^{-n(n+1)Ds'}] \quad (A.1)$$

Using the orthogonality relation of the spherical harmonics and after tedious calculations, we have

$$\omega_x^{(2)}(c,\phi,s) = \frac{1}{4\pi D^2}\left\{\left[\frac{1}{3}Ds - (1-e^{-2Ds})/4 + (1-e^{-6Ds})/36\right] + \frac{c}{2}e^{-2Ds}\right.$$

$$\cdot\left[\frac{3}{5}Ds - (1-\theta^{-4Ds})/4 + (1-e^{-10Ds})/25\right] + \frac{1}{2}\sum_{\ell=2}^{\infty}$$

$$\cdot P_\ell(c)e^{-\ell(\ell+1)Ds}\left[(2\ell+1)Ds - \frac{1-e^{-2(\ell+1)Ds}}{2(\ell+1)} + \frac{\ell}{2(2\ell-1)^2}\right.$$

$$\left.\cdot(1-e^{2(2\ell-1)Ds}) + \frac{e^{2\ell Ds}-1}{2\ell} + \frac{(\ell+1)(1-e^{-2(2\ell+3)Ds})}{2(2\ell+3)^2}\right]$$

$$+\frac{1}{2}\sum\frac{(\ell-2)!}{(\ell+2)!}P_\ell^2(c)\cos 2\phi\; e^{-\ell(\ell+1)Ds}$$

$$\cdot\left[\frac{3(\ell-1)(\ell+2)(2\ell+1)}{(2\ell-1)(2\ell+3)}Ds - \frac{\ell(\ell-1)}{2(\ell+1)}(1-e^{-2(\ell+1)Ds}) - \right.$$

$$- \frac{(\ell+1)(\ell+2)}{2\ell} (e^{2\ell Ds} - 1) + \frac{\ell(\ell+1)(\ell+2)}{2(2\ell-1)^2} (e^{2(2\ell-1)Ds} - 1)$$

$$- \frac{\ell(\ell+1)(\ell-1)}{(2\ell+3)^2} (1 - e^{-2(2\ell+3)Ds}) \Bigg] \Bigg\} \qquad (A.2)$$

2. For the general initial condition (53), it can be shown that

$$\omega_z^{(1)}(c,\phi,s) = \sum_{n=o}^{\infty} \sum_{m=-n}^{+n} \sqrt{\frac{(n+1-m)(n+1+m)}{(2n+1)(2n+3)}} \frac{1}{2(n+1)D}$$

$$\cdot \left[e^{-n(n+1)Ds} - e^{-n(n+1)(n+2)Ds}\right]$$

$$\cdot \left[A_{n+1,m} \; Y_n^m \; (c,\phi) + A_{nm} \; Y_{n+1}^m \; (c,\phi)\right] \qquad (A.3)$$

REFERENCES

Bremmer, H., "Random Volume Scattering," Radio Science, 68D, 967-981 (Sept., 1964).

Chandrasekhar, S., "Stochastic Problems in Physics and Astronomy," Rev. Mod. Phys., 15, 1-89 (Jan., 1943).

Chandrasekhar, S., Radiative Transfer, Dover Publications, Inc., 1960.

Chernov, L. A., "Wave Propagation in a Random Medium," p. 18, McGraw-Hill Book Co., 1960.

Keller, J. B., "Wave Propagation in Random Media," Proc. Sym. Appl. Math., 13, 227-246 (1962).

Komisarov, V. M., "Ray Statistics in a Plane Layer Medium with Random Inhomogeneity," Soviet Radiophysics, 9, 193-198 (March-April, 1966).

Komisarov, V. M., "Fluctuations in a Bounded Light Beam Propagating in a Randomly Inhomogeneous Medium," Soviet Physics JETP, 25, 467-469 (Sept., 1967).

Lyon, Richard H., "On the Diffusion of Sound Waves in a Turbulent Atmosphere," J. Acoust. Soc., 31, 1176-1182 (Sept., 1959).

Montgomery, D. C., and D. A. Tidman, Plasma Kinetic Theory, Chapter 2, McGraw-Hill Book Company, 1964.

Morse, P. M., and H. Feshbach, "Methods of Theoretical Physics," p. 467, McGraw-Hill Book Co., 1953.

Salpeter, E. E., and S. B. Treiman, "Multiple Scattering in the Diffusion Approximation," J. Math. Phys., 5, 659-668 (May, 1964).

So, K. C., "Generalized Stochastic Equations and Their Applications to Plasmas," Ionosphere Radio Laboratory, T. R. No. 31, Department of Electrical Engineering, University of Illinois, Urbana, Illinois (Sept., 1967).

Stratonovich, R. L., "Topics in the Theory of Random Noise," Gordon and Breach Science Publishers, Inc. (1963).